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# Schrödinger's Cataplex <sup>1</sup>

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## ABSTRACT

We discuss elementary entwiners that cross-weave the variables of certain integrable models: Liouville, sine-Gordon, and sinh-Gordon field theories in two-dimensional spacetime, and their quantum mechanical reductions. First we define a complex time parameter that varies from one energy-shell to another. Then we explain how field propagators can be simply expressed in terms of elementary functions through the combination of an evolution in this complex time and a duality transformation.

## IT'S COMPLEX TIME

One hundred years ago at the close of the 19<sup>th</sup> century, just before Planck's discovery of light quanta, H. M. Macdonald [21] considered the mathematical problem of determining zeroes of Bessel functions in the complex plane. He was led to find the lovely integral identity

$$K_\nu(e^x) K_\nu(e^y) = \int_{-\infty}^{+\infty} dz S(x, y, z) K_\nu(e^z) .$$

The kernel in the integral is a simple, symmetric exponential of exponentials.

$$S(x, y, z) = \frac{1}{2} \exp(-F(x, y, z)) , \quad F(x, y, z) = \frac{1}{2} (e^{x+y-z} + e^{x-y+z} + e^{-x+y+z}) .$$

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The most direct way to prove this result is through the use of the Heine-Schl fli identity that cogently expresses modified Bessel functions as an integral transform (cf. [32], §6.22 and §13.71).

$$K_\nu(e^x) = K_{-\nu}(e^x) = \frac{1}{2} \int_{-\infty}^{\infty} dX \exp(-e^x \cosh X + \nu X) .$$

When this is substituted for each of the Bessel functions in the previous bilinear, a simple change of integration variables immediately yields Macdonald's identity. (For another derivation, which overlaps with many of the standard textbook methods [23, 25] of obtaining similar integral relations for Mathieu functions, see [10].)

More recently in the 20<sup>th</sup> century, the modified Bessel functions in Macdonald's identity have appeared in a physical context as solutions of the Liouville quantum mechanics. For Liouville quantum mechanics the Hamiltonian is  $H = p^2 + e^{2x}$ . Coordinate space energy eigenfunctions are then solutions of

$$\left(-\frac{d^2}{dx^2} + e^{2x}\right) \psi_E(x) = E \psi_E(x) .$$

For  $0 < E < \infty$  the bounded solutions<sup>3</sup> are

$$\psi_E(x) = \psi_E^*(x) = \frac{1}{\pi} \sqrt{\sinh(\pi\sqrt{E})} K_{i\sqrt{E}}(e^x) .$$

As indicated, these  $\psi_E$ 's are real. For  $E = 0$  there is no solution [12]. For other values of  $E$  the wave functions are ortho-normalized such that  $\int_{-\infty}^{+\infty} dx \psi_{E_1}(x) \psi_{E_2}(x) = \delta(E_1 - E_2)$ . The wave functions are also complete on the appropriate space of bounded wave functions<sup>4</sup>, such that  $\int_0^\infty dE \psi_E(x) \psi_E(y) = \delta(x - y)$ .

From the reality and completeness of the Liouville wave functions for real  $x$ ,  $y$ , and  $z$ , it follows that another way to state Macdonald's identity<sup>5</sup> is

$$S(x, y, z) = \int_0^\infty dE K_{i\sqrt{E}}(e^z) \psi_E(x) \psi_E^*(y) .$$

Upon comparing this expression with the standard form for the propagator as a bilinear in wave functions,

$$G(x, y; t) = \int_0^\infty dE e^{-iEt} \psi_E(x) \psi_E^*(y) ,$$

a physical interpretation of Macdonald's identity is immediately apparent. Macdonald's kernel  $S(x, y, z)$  is precisely the Liouville propagator in the complex time plane, with the identification

$$e^{-iEt} = K_{i\sqrt{E}}(e^z) .$$

The parameters  $t$  and  $z$  are in direct correspondence. That is, the propagator may be written as

$$G(x, y; t) \doteq \frac{1}{2} \exp\left(-\frac{1}{2}(e^{x+y-z} + e^{x-y+z} + e^{-x+y+z})\right) ,$$

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<sup>3</sup>There are also unbounded solutions,  $I_\nu$ , for which Macdonald's integral relation has a "sister" identity:  $\theta(y-x) I_\nu(e^x) K_\nu(e^y) + \theta(x-y) I_\nu(e^y) K_\nu(e^x) = \int_{-\infty}^{+\infty} dz S(x, y, z) I_\nu(e^z)$ .

<sup>4</sup>A proof of completeness is given in [17], for example.

<sup>5</sup>The Liouville-to-free-particle transformation kernel is obtained by taking the limit of  $F$  as  $y, z \rightarrow -\infty$ , with  $x$  and  $X \equiv y - z$  fixed. This gives  $F(x, y, z) \rightarrow e^x \cosh X$ , with the variable  $X$  acting as the free particle coordinate in the Heine-Schl fli transform previously noted.

where  $\doteq$  signifies equality on a given energy shell for which  $t = \frac{i}{E} \ln \left( K_{i\sqrt{E}}(e^z) \right)$ . Real  $z$  corresponds to complex  $t$ .

This elementary but nontrivial form for the propagator has the virtue of having explicitly simple  $x$  and  $y$  coordinate dependence, without being either pathological or tautological<sup>6</sup>. While the time dependence is somewhat mysteriously encoded in the variable  $z$ , the coordinate dependence is quite transparent. In principle, such explicit coordinate dependence for *any* propagator should greatly facilitate extracting the coordinate dependence of the corresponding energy eigenstates. Recall that two general methods for extracting such information from the propagator are either to project onto a particular energy by Fourier transforming in the time (that is, construct the Green function and examine the residues of its poles), or to take the deep Euclidean time limit (and, say, isolate a particular exponentially decaying term). Similar general methods should be possible<sup>7</sup> involving the variable  $z$ .

Some care is required, however, since the propagator interpretation of Macdonald's identity implies that the relation between the variable  $z$  and the time  $t$  is energy dependent, and not just through the combination  $Et$ . For example, for large  $|e^z|$ , so long as  $|\arg(e^z)| < \frac{3}{2}\pi$ , the asymptotic behavior of the modified Bessel function is

$$K_\nu(e^z) \sim \sqrt{\frac{\pi}{2e^z}} e^{-e^z} \left\{ 1 + \frac{4\nu^2 - 1}{8} e^{-z} + \frac{1}{2!} \frac{4\nu^2 - 1}{8} \frac{4\nu^2 - 9}{8} e^{-2z} + \dots \right\}.$$

This means, for deep Euclidean time  $t = -iT$ ,  $T \rightarrow \infty$ ,  $e^{-iEt} \sim e^{-e^z - \frac{1}{2}z + \ln \sqrt{\frac{\pi}{2}} + \ln\{1 + \dots\}}$ . That is

$$T \sim \frac{1}{E} \left( e^z + \frac{1}{2}z - \ln \sqrt{\frac{\pi}{2}} + \frac{1 + 4E}{8} e^{-z} + O(e^{-2z}) \right).$$

Curves in the complex  $z$  plane which correspond to real time evolution would be contours of constant modulus for  $K_{i\sqrt{E}}(e^z)$ . If these are open contours, the corresponding time evolution would be over an interval, perhaps infinite. (If these are simple closed contours, the corresponding time variable would be periodic, and the contours might therefore be appropriate to describe Liouville quantum mechanics on closed time-like curves, or perhaps at finite temperature, in the extension to Liouville and other field theories in the following.)

Also note that this relation between  $z$  and  $t$  is not as strange as it might first appear. Analytic continuation of the time variable is of course a standard practice, but in addition, simple energy dependent (or more generally "state" or "system" dependent) redefinitions of the time variable are standard techniques in several areas of physics. For an ancient example, in celestial mechanics the use of the various "anomaly" variables (such as the mean anomaly  $l(t) = a^{-3/2}t$  where  $a$  is the semi-major axis for a particular orbit [1]) amounts to a somewhat trivial choice of system-dependent time. For a more contemporary class of examples, general relativists routinely use coordinates in which the new time variable depends on both the old space and time coordinates as well as on the properties of the system in question (e.g. black hole Kerr coordinates [31]).

It seems odd to me that this interpretation of Macdonald's identity as a propagator has escaped notice until now. Nonetheless, it would appear that this simple interpretation has not been appreciated previously. For instance, there is no mention of it among the various propagators compiled in [18], although the authors of that compilation did indeed use the previously mentioned sister identity to recast the path integral form of the propagator into the standard sum over wave

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<sup>6</sup>By "tautological form" we mean  $G(x, y; t) \doteq \delta(x - y) \exp(-iwt)$ , where  $w = tE$  on a given energy shell. This is a true statement, but by itself it clearly does not represent *any* progress in determining the explicit properties of a system. Although for at least one simple case it does lead to the familiar form for the propagator. If it is applied as a kernel in an integral transform of a free particle wave function, an integration by parts immediately produces the well-known result:  $\exp(it\partial^2/\partial x^2) \delta(x - y) = \frac{1}{\sqrt{4\pi it}} \exp(i(x - y)^2/4t)$ .

<sup>7</sup>In particular, another way to think of Macdonald's original identity is just as a means of extracting the residues of the Liouville Green function poles.

function bilinears [17]. As far as I can tell, the  $t \leftrightarrow z$  correspondence is not realized either in the work of Anderson, et al., [2, 3, 4]. Finally, in their recent analysis [10] Davis and Ghandour also do not make any connection to the propagator<sup>8</sup>. All in all, it would seem that a more careful and thorough analysis of the  $t \leftrightarrow z$  correspondence is warranted.

## DUALITY ENTWINES

When this correspondence is extended to the field theory case, a more compelling collection of ideas emerges. In 1+1 field theory, a correspondence between  $z$  and complex time also exists when conjoined with a duality transformation. This follows from replacing the  $x$  and  $y$  variables in the exponentials of Macdonald's kernel with local fields  $\phi(\sigma)$  and  $\psi(\sigma)$ , integrating over the spatial coordinate  $\sigma$ , and adding the elementary duality generator  $\int \phi \partial_\sigma \psi d\sigma$  as follows<sup>9</sup>.

$$\mathfrak{S}(\phi, \psi, z) \equiv \exp i \int d\sigma \left( \phi \partial_\sigma \psi + \frac{1}{2} \left( e^{\phi+\psi-z} - e^{\phi-\psi+z} - e^{-\phi+\psi+z} \right) \right).$$

We have also adjusted the phases in the exponential to conform to those in the usual Schrödinger equation, shifting  $z \rightarrow z + i\pi/2$ .

The kernel  $\mathfrak{S}$  “entwines” or “cross-weaves” various field theory operators for the  $\phi$  and  $\psi$  fields within the Schrödinger wave-functional framework, as we now explain. We first observe that

$$\begin{aligned} -i \frac{\delta}{\delta \phi(\rho)} \mathfrak{S}(\phi, \psi, z) &= \left( \partial_\rho \psi + \frac{1}{2} \left( e^{\phi+\psi-z} - e^{\phi-\psi+z} + e^{-\phi+\psi+z} \right) \right) \mathfrak{S}(\phi, \psi, z), \\ -i \frac{\delta}{\delta \psi(\rho)} \mathfrak{S}(\phi, \psi, z) &= \left( -\partial_\rho \phi + \frac{1}{2} \left( e^{\phi+\psi-z} + e^{\phi-\psi+z} - e^{-\phi+\psi+z} \right) \right) \mathfrak{S}(\phi, \psi, z), \end{aligned}$$

where we have assumed that  $\frac{\delta}{\delta \psi}(\phi\psi) = 0$  at the ends of the  $\sigma$  integration range. These results immediately allow us to show that the kernel relates local momentum density operators for the two fields.

By definition these momentum operators are  $\mathcal{P}_\phi(\rho) = -i\partial_\rho \phi \frac{\delta}{\delta \phi(\rho)}$  and  $\mathcal{P}_\psi(\rho) = -i\partial_\rho \psi \frac{\delta}{\delta \psi(\rho)}$ , so that

$$\begin{aligned} \mathcal{P}_\phi(\rho) \mathfrak{S}(\phi, \psi, z) &= \partial_\rho \phi \left( \partial_\rho \psi + \frac{1}{2} \left( e^{\phi+\psi-z} - e^{\phi-\psi+z} + e^{-\phi+\psi+z} \right) \right) \mathfrak{S}(\phi, \psi, z), \\ \mathcal{P}_\psi(\rho) \mathfrak{S}(\phi, \psi, z) &= \partial_\rho \psi \left( -\partial_\rho \phi + \frac{1}{2} \left( e^{\phi+\psi-z} + e^{\phi-\psi+z} - e^{-\phi+\psi+z} \right) \right) \mathfrak{S}(\phi, \psi, z). \end{aligned}$$

Combining these and assuming that  $z$  is *not* also a local field, so that  $\partial_\rho z = 0$ , we obtain

$$(\mathcal{P}_\phi(\rho) + \mathcal{P}_\psi(\rho)) \mathfrak{S}(\phi, \psi, z) = \mathfrak{S}(\phi, \psi, z) i\partial_\rho F(\phi, \psi, z + i\pi/2).$$

We now integrate this last equation over  $\rho$  to obtain the total, global momentum operators for the two fields:  $\mathbf{P}_\phi = \int d\rho \mathcal{P}_\phi(\rho)$ ,  $\mathbf{P}_\psi = \int d\rho \mathcal{P}_\psi(\rho)$ .

If we impose boundary conditions in  $\rho$  such that  $0 = \int d\rho \partial_\rho F(\phi, \psi, z + i\pi/2)$ , then we have

$$\mathbf{P}_\phi \mathfrak{S}(\phi, \psi, z) = -\mathbf{P}_\psi \mathfrak{S}(\phi, \psi, z).$$

<sup>8</sup>Not even when they treat a special limiting case where the system under study reduces to the simple harmonic oscillator!

<sup>9</sup>The Liouville-to-free-field transformation kernel is obtained by taking the limit as  $\psi, z \rightarrow -\infty$ , with  $\phi$  and  $\varphi \equiv \psi - z$  fixed. This gives  $\mathfrak{S}(\phi, \psi, z) \rightarrow \exp i \int d\sigma (\phi \partial_\sigma \varphi + e^\phi \sinh \varphi)$ , with the variable  $\varphi$  acting as the free field. The corresponding functional integral transforms discussed below become in this limit those first used in [7] to compute certain Liouville energy eigenfunctional matrix elements.

Using functional integration by parts, this implies that the two momenta are exchanged by, or entwined with the kernel in an integral transform. That is, the momenta of  $\phi$  and  $\psi$  wave functionals become cross-weaved with one another when these functionals are related through the use of  $\mathfrak{S}(\phi, \psi, z)$  in a functional integral transform. More explicitly, let<sup>10</sup>

$$\Phi(\phi) = \int D\psi \mathfrak{S}(\phi, \psi, z) \Psi(\psi) .$$

Then

$$\mathbf{P}_\phi \Phi(\phi) = \int D\psi \mathbf{P}_\phi \mathfrak{S}(\phi, \psi, z) \Psi(\psi) = \int D\psi \mathfrak{S}(\phi, \psi, z) \mathbf{P}_\psi \Psi(\psi) ,$$

where we have discarded the surface terms arising from the functional integration by parts.

This last relation is precisely what we mean by the kernel entwining or cross-weaving the momentum operators in the Schrödinger functional framework<sup>11</sup>.

A perspicacious observer would notice that the momenta of  $\phi$  and  $\psi$  theories are entwined in precisely the same way for any choice  $\mathfrak{S} = \exp i \int d\sigma (\phi \partial_\sigma \psi + f(\phi, \psi))$  regardless of the form of  $f(\phi, \psi)$ . The crucial behavior is provided solely by the elementary duality generator  $\int d\sigma \phi \partial_\sigma \psi$ , which by itself would interchange spatial derivatives of the fields with their canonically conjugate variables (realized as functional derivatives here) just as in the classical theory. So, cross-weaving spatial momentum operators inside the functional integral transform is a relatively trivial task that places only minor restrictions on the kernel. In a much less trivial way, the previous kernel  $\mathfrak{S}(\phi, \psi, z)$  also entwines with the energy operators for the  $\phi$  and  $\psi$  fields.

To demonstrate this, we need to take second functional derivatives. Rewrite the previous first functional derivatives as

$$\begin{aligned} -i \left( \frac{\delta}{\delta \phi(\rho)} + \frac{\delta}{\delta \psi(\rho)} \right) \mathfrak{S}(\phi, \psi, z) &= \left\{ \partial_\rho (\psi - \phi) + e^{-z+\phi(\rho)+\psi(\rho)} \right\} \mathfrak{S}(\phi, \psi, z) , \\ -i \left( \frac{\delta}{\delta \phi(\rho)} - \frac{\delta}{\delta \psi(\rho)} \right) \mathfrak{S}(\phi, \psi, z) &= \left\{ \partial_\rho (\psi + \phi) + e^z \left( e^{-\phi(\rho)+\psi(\rho)} - e^{\phi(\rho)-\psi(\rho)} \right) \right\} \mathfrak{S}(\phi, \psi, z) , \end{aligned}$$

and combine these to get the second derivative

$$\begin{aligned} (-i)^2 \left( \frac{\delta}{\delta \phi(\rho_1)} - \frac{\delta}{\delta \psi(\rho_1)} \right) \left( \frac{\delta}{\delta \phi(\rho_2)} + \frac{\delta}{\delta \psi(\rho_2)} \right) \mathfrak{S}(\phi, \psi, z) &= \{ 2i \partial_{\rho_2} \delta(\rho_2 - \rho_1) \} \mathfrak{S}(\phi, \psi, z) \\ + \left\{ \partial_{\rho_2} (\psi - \phi) + e^{-z+\phi(\rho_2)+\psi(\rho_2)} \right\} &\left\{ \partial_{\rho_1} (\psi + \phi) + e^z \left( e^{-\phi(\rho_1)+\psi(\rho_1)} - e^{\phi(\rho_1)-\psi(\rho_1)} \right) \right\} \mathfrak{S}(\phi, \psi, z) . \end{aligned}$$

<sup>10</sup>Recall from above real  $z$  corresponds to Euclidean time. So for arbitrary  $z$  this transformation does not necessarily preserve the wave-functional normalizations, and hence is a similarity transformation, but not necessarily a unitary one [6]. Nonetheless, this does not effect the present discussion.

<sup>11</sup>Hence the title of this talk. From Liddell-Scott-Jones Lexicon of Classical Greek: *καταπλεκω* (kataplekô) - *entwine, plait*. Or from Herodotus *Histories* (Loeb) [3.98.4]:

Houtoi men dê tôn Indôn phoreousi esthêta phloĩnên: epean ek tou potamou phloun amêsôsi kai kopsôsi,  
to entheuten phormou tropon **kataplexantes** hôs thôrêka endunousi.

(These Indians wear clothes of bullrushes; they mow and cut these from the river, then **having woven them crosswise** like a mat, wear them like a breastplate.)

Note that kataplexantes is the plural of the active participle of kataplekô, which we have chosen to distill for obvious reasons to a more contemporary “cataplex”. Of course, as scholars of classical Greek will note, there is also: *καταπληξ* (kata-plêx) - *stricken, struck*, usu. metaph., *stricken with amazement, astounded*. However, this too is an appropriate meaning for the situation under discussion, in our opinion.

After giving this talk, we learned that M. Gell-Mann had previously drawn on the Indo-European root \*plek-, from which *πλεκω* derives, to introduce the term *plectics* for “a broad transdisciplinary subject covering aspects of simplicity and complexity as well as the properties of complex adaptive systems”. (cf. <http://www.santafe.edu/sfi/People/mgm/plectics.html>)

Taking the  $1 \longleftrightarrow 2$  symmetric,  $\rho_1 \rightarrow \rho$ ,  $\rho_2 \rightarrow \rho$  limit of this gives

$$\begin{aligned}
& (-i)^2 \left( \frac{\delta^2}{\delta\phi(\rho)^2} - \frac{\delta^2}{\delta\psi(\rho)^2} \right) \mathfrak{S}(\phi, \psi, z) \\
& \equiv \frac{1}{2} \lim_{\rho_1, \rho_2 \rightarrow \rho} (-i)^2 \left( \frac{\delta}{\delta\phi(\rho_1)} - \frac{\delta}{\delta\psi(\rho_1)} \right) \left( \frac{\delta}{\delta\phi(\rho_2)} + \frac{\delta}{\delta\psi(\rho_2)} \right) \mathfrak{S}(\phi, \psi, z) + (1 \longleftrightarrow 2) \\
& = \left\{ \partial_\rho(\psi - \phi) + e^{-z+\phi(\rho)+\psi(\rho)} \right\} \left\{ \partial_\rho(\psi + \phi) + e^z \left( e^{-\phi(\rho)+\psi(\rho)} - e^{\phi(\rho)-\psi(\rho)} \right) \right\} \mathfrak{S}(\phi, \psi, z) \\
& = \left\{ (\partial_\rho\psi)^2 - (\partial_\rho\phi)^2 + e^{2\psi(\rho)} - e^{2\phi(\rho)} \right\} \mathfrak{S}(\phi, \psi, z) + 2\mathfrak{S}(\phi, \psi, z) \partial_\rho F(\phi, \psi, z) .
\end{aligned}$$

Now the local energy density operators for the  $\phi$  and  $\psi$  fields are of the same form for either:

$$\mathcal{H}_\phi(\rho) = -\frac{1}{2} \frac{\delta^2}{\delta\phi(\rho)^2} + \frac{1}{2} (\partial_\rho\phi)^2 + \frac{1}{2} e^{2\phi(\rho)} , \quad \mathcal{H}_\psi(\rho) = -\frac{1}{2} \frac{\delta^2}{\delta\psi(\rho)^2} + \frac{1}{2} (\partial_\rho\psi)^2 + \frac{1}{2} e^{2\psi(\rho)} .$$

In view of these, the previous second derivative relation is

$$\mathcal{H}_\phi(\rho) \mathfrak{S}(\phi, \psi, z) = \mathcal{H}_\psi(\rho) \mathfrak{S}(\phi, \psi, z) + \mathfrak{S}(\phi, \psi, z) \partial_\rho F(\phi, \psi, z) .$$

If we once again integrate over  $\rho$  to obtain the total energy operators for either field,  $\mathbf{H}_\phi = \int d\rho \mathcal{H}_\phi(\rho)$  and  $\mathbf{H}_\psi = \int d\rho \mathcal{H}_\psi(\rho)$ , and if we again impose boundary conditions in  $\rho$  such that  $0 = \int d\rho \partial_\rho F(\phi, \psi, z)$ , we finally obtain

$$\mathbf{H}_\phi \mathfrak{S}(\phi, \psi, z) = \mathbf{H}_\psi \mathfrak{S}(\phi, \psi, z) .$$

Acting on wave functionals, this leads to

$$\mathbf{H}_\phi \Phi(\phi) = \int D\psi \mathbf{H}_\phi \mathfrak{S}(\phi, \psi, z) \Psi(\psi) = \int D\psi \mathfrak{S}(\phi, \psi, z) \mathbf{H}_\psi \Psi(\psi) .$$

The energy operators are therefore also cross-woven by the kernel  $\mathfrak{S}(\phi, \psi, z)$ .

All this entwined structure is present in other models besides the Liouville. For example, the sinh-Gordon and sine-Gordon theories in  $1+1$  dimensions also have simple entwining kernels<sup>12</sup> explicitly given by exponentials of exponentials. This is not surprising. These models are well-known [24, 30] to be the only ones (besides quadratic Hamiltonians) involving one-component fields which can be reduced to first-order differential equations involving two such fields whose consistency requires the same second-order equations for either field separately (i.e. auto-Bäcklund transformations, as discussed in [28]). In that purely classical context, the parameter  $z$  above is known as a “Bäcklund parameter”. Exponentiation of the corresponding classical generators of the first-order equations, to arrive at quantum theories, follows from Dirac’s correspondence rule [13]. The fact that the naive correspondence works exactly, without the need for local quantum corrections, for the Liouville, sinh-Gordon, and sine-Gordon theories, is in our view the essence of the integrability of these models<sup>13</sup>. However, it should be stressed that the exact propagator for *any* theory provides an exact entwining kernel<sup>14</sup>, even if that propagator has extensive quantum

<sup>12</sup>I obtained the kernels for the sine-Gordon and sinh-Gordon theories in the 1980’s, after having spent a few years working on the quantization of the classical Bäcklund transformation connecting Liouville and free fields, initially in an operator framework [5] and subsequently using functional methods in collaboration with Ghassan Ghandour and my student Thomas McCarty. The latter work was not published until 1991 [7, 22].

<sup>13</sup>Davis and Ghandour [10] have effectively shown that there are no other models involving one-component fields for which the classical generators can be used in Dirac’s correspondence to obtain valid kernels. If more components are allowed, however, there are many more models such that the classical generators provide entwining kernels. The  $\sigma$  model is one such notable example [9].

<sup>14</sup>We leave it as a straightforward exercise for the student to show that the usual propagators for the linear potential and the harmonic oscillator are indeed entwiners in the above sense, and for these simple QM examples the correspondence between  $t$  and  $z$  is in fact energy *independent*.

corrections, and even if those corrections are non-local. This point has been noted previously [7] with somewhat different emphasis.

Let us summarize the results for exponential potentials. The general form of the kernel is:

$$\mathfrak{S}(\phi, \psi, z) \equiv \exp i \int d\sigma \mathfrak{F}(\phi(\sigma), \psi(\sigma), z)$$

The explicit forms of the generators, the local energy densities, and their effects on the kernel are in the following table.

Liouville	$\mathcal{H}_\phi = -\frac{1}{2} \frac{\delta^2}{\delta\phi(\rho)^2} + \frac{1}{2} (\partial_\rho \phi)^2 + \frac{1}{2} e^{2\phi(\rho)}$ $\mathfrak{F} \equiv \phi \partial_\sigma \psi + \frac{1}{2} (e^{-z+\phi+\psi} - e^{z-\phi+\psi} - e^{z+\phi-\psi})$ $\mathcal{H}_\phi \mathfrak{S} = \mathcal{H}_\psi \mathfrak{S} + \frac{1}{2} \mathfrak{S} \partial_\rho \{e^{-z+\phi+\psi} + e^{z-\phi+\psi} + e^{z+\phi-\psi}\}$
sinh-Gordon	$\mathcal{H}_\phi = -\frac{1}{2} \frac{\delta^2}{\delta\phi(\rho)^2} + \frac{1}{2} (\partial_\rho \phi)^2 + \cosh 2\phi(\rho)$ $\mathfrak{F} \equiv \phi \partial_\sigma \psi + e^{-z} \cosh(\phi + \psi) - e^{+z} \cosh(\phi - \psi)$ $\mathcal{H}_\phi \mathfrak{S} = \mathcal{H}_\psi \mathfrak{S} + \mathfrak{S} \partial_\rho \{e^{-z} \cosh(\phi + \psi) + e^{+z} \cosh(\phi - \psi)\}$
sine-Gordon	$\mathcal{H}_\phi = -\frac{1}{2} \frac{\delta^2}{\delta\phi(\rho)^2} + \frac{1}{2} (\partial_\rho \phi)^2 - \cos 2\phi(\rho)$ $\mathfrak{F} \equiv \phi \partial_\sigma \psi - e^{-z} \cos(\phi + \psi) + e^{+z} \cos(\phi - \psi)$ $\mathcal{H}_\phi \mathfrak{S} = \mathcal{H}_\psi \mathfrak{S} + \mathfrak{S} \partial_\rho \{-e^{-z} \cos(\phi + \psi) - e^{+z} \cos(\phi - \psi)\}$

The above generators for the sinh-Gordon and sine-Gordon theories have been used in a classical context for a long time<sup>15</sup>. They appear in textbooks as the generators of (auto) Bäcklund transformations [28, 33], where for the sine-Gordon case their functional derivatives are most often employed to generate classical  $(N+1)$ -soliton solutions starting from  $N$ -soliton solutions, with  $\phi = 0$  as the trivial  $N = 0$  soliton [29, 19]. In that classical situation the Bäcklund parameter  $z$  is related to the rapidity of the soliton's center of mass.

Now let us reconsider those total divergence terms that are produced by entwining the densities with  $\mathfrak{S}(\phi, \psi, z)$  and show that they are just the usual conformal improvements for the energy-momentum tensor. We find in all three cases:

$$\left( \mathcal{H}_\phi - \partial_\rho^2 \phi(\rho) + \mathcal{P}_\phi - \partial_\rho \left( -i \frac{\delta}{\delta\phi(\rho)} \right) \right) \mathfrak{S} = \left( \mathcal{H}_\psi + \partial_\rho \left( -i \frac{\delta}{\delta\psi(\rho)} \right) - \mathcal{P}_\psi - \partial_\rho^2 \psi(\rho) \right) \mathfrak{S}$$

$$\left( \mathcal{H}_\phi - \partial_\rho^2 \phi(\rho) - \mathcal{P}_\phi + \partial_\rho \left( -i \frac{\delta}{\delta\phi(\rho)} \right) \right) \mathfrak{S} = \left( \mathcal{H}_\psi + \partial_\rho \left( -i \frac{\delta}{\delta\psi(\rho)} \right) + \mathcal{P}_\psi + \partial_\rho^2 \psi(\rho) \right) \mathfrak{S}$$

The pattern clearly shows that the densities for the two fields undergo dual improvements.

$$\mathcal{H}_\phi \rightarrow \mathcal{H}_\phi - \partial_\rho^2 \phi(\rho) \ , \quad \mathcal{P}_\phi \rightarrow \mathcal{P}_\phi - \partial_\rho \left( -i \frac{\delta}{\delta\phi(\rho)} \right) \ ,$$

<sup>15</sup>The Liouville generator follows from the sinh-Gordon generator as a contraction: shift  $\phi \rightarrow \phi + w$ ,  $\psi \rightarrow \psi + w$ ,  $z \rightarrow z + w$ , rescale  $\sigma \rightarrow e^{-w}\sigma$ , and take  $w \rightarrow \infty$ .

$$\mathcal{H}_\psi \rightarrow \mathcal{H}_\psi + \partial_\rho \left( -i \frac{\delta}{\delta \psi(\rho)} \right), \quad \mathcal{P}_\psi \rightarrow \mathcal{P}_\psi + \partial_\rho^2 \psi(\rho).$$

Expressing this covariantly for classical densities, using  $T_{\mu\nu}$  for the conventional unmodified energy-momentum tensor and  $\theta_{\mu\nu}$  for the conformally improved one, we would have the on-shell relations

$$\begin{aligned} \theta_{\mu\nu}(\phi) &= T_{\mu\nu}(\phi) + (g_{\mu\nu} \square - \partial_\mu \partial_\nu) \phi = T_{\mu\nu}(\phi) + \varepsilon_{\mu\alpha} \varepsilon_{\nu\beta} \partial^\alpha \partial^\beta \phi \\ \theta_{\mu\nu}(\psi) &= T_{\mu\nu}(\psi) + \frac{1}{2} (\varepsilon_{\mu\alpha} \partial^\alpha \partial_\nu + \varepsilon_{\nu\alpha} \partial^\alpha \partial_\mu) \psi \end{aligned}$$

Note that  $\varepsilon_{0\alpha} \partial^\alpha f = \partial_\sigma f$  and  $\varepsilon_{1\alpha} \partial^\alpha f = \partial_\tau f$  where  $\sigma$  and  $\tau$  are the space and time coordinates in  $1+1$  dimensions.

So, with these local modifications, the energy and momentum densities are entwined without left-over total derivatives. In the case of the Liouville theory, at least, this means that the full Virasoro algebra entwines with  $\mathfrak{S}$ . For an earlier quantum mechanical example of this situation, see [6].

## WISHFUL THOUGHTS

Perhaps other models can be found to have simple propagators using the approach discussed here. An interesting case would be the nonlinear  $\sigma$  model, and its supersymmetric siblings, which can be entwined with a dual  $\sigma$  model at the expense of deforming the field manifold and introducing torsion [9, 8]. It is not yet known how to incorporate the parameter  $z$ , and by correspondence the time, into this transformation.

Perhaps this approach to propagators is also useful when the  $(\tau, \sigma) = (\zeta^0, \zeta^1)$  manifold is not intrinsically flat. For example, classical relations *at fixed time* between the Liouville field  $\phi$  and a “free” field  $\psi$  have been discussed before [27] (also see [11]). These fields satisfy

$$D^\mu D_\mu \phi = \frac{1}{2g} R - \frac{4m^2}{g} e^{2g\phi}, \quad D^\mu (\partial_\mu \psi - \frac{1}{g} \omega_\mu) = 0,$$

where zweibein  $e_\mu^a$ , connection  $\omega_\mu = \eta_{ab} e_\mu^a \epsilon^{\nu\lambda} \partial_\nu e_\lambda^b$ , and scalar curvature  $R = -2\epsilon^{\mu\nu} \partial_\mu \omega_\nu$  are given functions that depend on  $(\zeta^0, \zeta^1)$ . Canonical equivalence of the  $\phi$  and  $\psi$  fields in this curved surface situation again follows from a generating function, which here depends explicitly on  $\zeta^0$  even before evolution is included.

$$F[\phi, \psi; \zeta^0] = \int d\zeta^1 \left( \phi \left( \partial_1 \psi - \frac{1}{g} \omega_1(\zeta) \right) - \frac{2m}{g^2} e^{g\phi} e_1^a(\zeta) V_a(\psi) \right).$$

The tangent space vector  $V$  is given by  $(V_0, V_1) = (\cosh(g\psi), \sinh(g\psi))$ .

To generalize these decade-old results and find the analogue of MacDonald’s century-old propagator, we seek a generating function that yields exponential potentials for both fields, and that allows for evolution in  $\zeta^0$  through an explicit  $z$  parameter. Indeed, such a generalization is needed to make contact with other studies of propagators and correlation functions for the Liouville and sine/sinh-Gordon models, since these other studies almost invariably consider the underlying space-time to be a sphere. For example, see [16, 14, 34, 20, 15, 26]. It remains to entwine all these investigations.

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